Summary: The families of nets of quadrics in $\mathbf{P}^5$ whose base locus is isomorphic to the associated sextic double plane

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Contents

1 Preliminaries

The title of this thesis is perhaps a little bit restrictive with respect to the generality of arguments which has been afforded. Nevertheless it is representative of the original motivation for most of the work contained here and for most of the results. The more general theme, which is connecting the various sections, concerns K3 surfaces. It has to do with the following, certainly too vague and indeterminate, question:

PROBLEM. *Let S be a K3 surface: under which conditions on S some geometrical objects, which are naturally associated to S, are birational to a symmetric product of S?*

We recall that a K3 surface S is a compact complex surface which is simply connected and whose first Chern class is trivial. We will only deal with *projective* K3 surfaces i.e. with those K3 surfaces which can be embedded in a complex projective space. When speaking of a K3 surface, we will assume implicitely this special property.
Usually, the objects we are interested to will be depending on S and on some additional data. In particular we will need to fix a line bundle H on S which is usually very ample. The map associated to H defines an embedding

$$S \subset \mathbf{P}^g.$$

In such a situation it turns out that the surface S has specially nice projective properties: its hyperplane sections are canonical curves of genus g. Moreover, up to some precise exceptions, the homogeneous ideal of S is generated by the quadratic forms vanishing on S. In other words S is the base locus of the linear system

$$Q_S$$

of all quadrics containing S. Let us briefly introduce some of the geometric objects one can associate to S:

(1) The linear system Q_S is stratified by the loci

$$Q_S^k \ = \ \{ \ q \ \in \ Q_S \mid \text{rank}\ q \leq k \ \}.$$

If k is even there exists a natural double covering

$$\pi_k : \tilde{Q}_S^k \to Q_S^k,$$

branched along Q_S^{k-1}. We will be specially interested to the variety $\tilde{Q}_S^6$.

(2) Some other natural objects associated to S are certainly the moduli spaces

$$M_S(r; c_1, c_2)$$

of semistable sheaves on S having rank r and fixed Chern classes c_1, c_2. We will be interested to the case $r = 2$, specially in the following situations:
– when $M_S(r; c_1, c_2)$ is another compact K3 surface,
– when $\frac{1}{2}c_1^2 = c_2$ and $c_1 = H$.

(3) Finally, we mention some other K3 surfaces which are naturally defined by the transcendental part of the cohomology of S.

We recall that $H^2(S, \mathbf{Z})$ is a free abelian group of rank 22. $H^2(S, \mathbf{Z})$, endowed with the cup-product, is a lattice. As a lattice $H^2(S, \mathbf{Z})$ is isometric to the orthogonal sum

$$\Lambda =: U \perp U \perp U \perp E_8 \perp E_8,$$

where U and E_8 are the hyperbolic lattice and the unique even unimodular positive definite lattice of rank 8 (see [1]), respectively. We can fix an isometry

$$\psi : H^2(S, \mathbf{Z}) \to \Lambda.$$

Then we can consider the Hodge decomposition

$$H^2(S, \mathbf{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}.$$

$H^{2,0}$ is a 1–dimensional vector space, which is generated by the unique (up to scalars) non zero holomorphic 2–form on S. The image of $H^{2,0}$ by the isomorphism

$$\psi \otimes \mathbf{C} : H^2(S, \mathbf{C}) \to \Lambda \otimes_{\mathbf{Z}} \mathbf{C}$$

defines a point p_S in the projective space

$$\mathbf{P}^{21} = \mathbf{P}(\Lambda \otimes_{\mathbf{Z}} \mathbf{C}).$$

On the other hand, the orthogonal group $O(\Lambda)$ of all the isometries of Λ naturally acts on $\mathbf{P}^{21}$. By definition *the period of* S, is the $O(\Lambda)$–orbit of p_S. This definition is valid for any, even non algebraic, K3 surface. Next, let us consider two K3 surfaces S and S'. One of the versions of the theorem of Torelli for K3 surfaces says the following:
– *If S and S' have the same period then S and S' are biholomorphic.*

In the algebraic case one can give a natural variant of the previous construction. Let

$$NS(S) \subset H^2(S, \mathbf{Z})$$

be the Néron-Severi lattice of S. By definition the transcendental lattice of S is the orthogonal lattice

$$T(S) = NS(S)^{\perp}.$$

It is clear that $H^{2,0} \subset T(S) \otimes_{\mathbf{Z}} \mathbf{C}$. We can consider the family of all K3 surfaces S such that $T(S)$ is isometric to some abstract lattice T. Then, with exactly the same construction as above, we can fix an isometry $\psi : T(S) \to T$ and define a point $t(S)$ in the projective space

$$\mathbf{P}(T \otimes_{\mathbf{Z}} \mathbf{C}).$$

By definition, the orbit of $t(S)$ under the action of the orthogonal group $O(T)$, is *the transcendental period of* S. It is no longer true that, if S and S' have the same transcendental period, then they are biholomorphic. This motivates the following:

DEFINITION. *A K3 surface S' is a partner of S if S and S' have isometric transcendental lattices and the same transcendental period.*

The problem of describing the partners of S and counting their number has attracted much attention in the recent literature, (cfr. [2] [4] [14] [16] [3], for example).
Note that S and S' are partners if and only if there exists an isometry

$$\phi : T(S) \to T(S')$$

such that $\phi_{\mathbf{C}} : T(S) \otimes_{\mathbf{Z}} \mathbf{C} \to T(S') \otimes_{\mathbf{Z}} \mathbf{C}$ respects the Hodge decomposition. Following Mukai we will say that S and S' are *isogenous* iff there exists an isometry

$$\phi : T(S) \otimes_{\mathbf{Z}} \mathbf{Q} \to T(S') \otimes_{\mathbf{Z}} \mathbf{Q}$$

such that $\phi_{\mathbf{C}} : T(S) \otimes_{\mathbf{Z}} \mathbf{C} \to T(S') \otimes_{\mathbf{Z}} \mathbf{C}$ respects the Hodge decomposition.

The objects we have mentioned so far are of different nature. Nevertheless they are quite strictly related and, of course, they are related to the original K3 surface S. Before of further comments we can state some of the problems we have considered:

PROBLEM (1). *Under which conditions $S[n]$ is birational to $\tilde{Q}^6_S$ for some n?*

Here $S[n]$ denotes the Hilbert scheme of 0-dimensional subschemes of S having length n. In particular $S[n]$ *is a smooth desingularization of the n–symmetric product of S.* As Tjurin shows, there exists a natural birational map

$$M_S(2; c_1, c_2) \to \tilde{Q}^6_S$$

if $c_2 = \frac{1}{2}c_1^2$. Therefore Problem 1 is part of the following more general question:

PROBLEM (2). *Under which conditions $M_S(2; c_1, c_2)$ is birational to $S[n]$ for some n?*

Often, the answer to this question is 'no conditions'. For instance it is well known that

$$c_2 >> c_1^2 \implies M_S(2; c_1, c_2) \;\cong\; S[n],$$

for some n. However the previous statement (and its refinements: see [5] [13]) does not include the case $c_2 = \frac{1}{2}c_1^2$. In such a case we expect that, in the moduli space

$$\mathcal{K}_g$$

of pairs (S, H), there exists a countable union

$$D \subset \mathcal{K}_g$$

of divisors parametrizing all the pairs (S, H) such that $M_S(2; H, c_2)$ is birational to $S[n]$. Let us also recall that every space $M_S(r; c_1, c_2)$ is a deformation of $S[n]$ for some n. Therefore it is quite natural to ask about the loci in $\mathcal{K}_g$ where $M_S(r; c_1, c_2)$ is $S[n]$. For $g = 5$ we have the:

PROBLEM (3). *Under which conditions a partner of S becomes isomorphic to S?*

More precisely we can consider the previous moduli space $\mathcal{K}_g$ and we can look for the loci in $\mathcal{K}_g$ where some of the partners of S becomes isomorphic to S. One can also refine the problem in many ways. In particular one can replace $\mathcal{K}_g$ by the moduli space $\mathcal{K}_T$ of K3–surfaces with fixed transcendental lattice T.

2 The main problem ($g = 5$).

The present work is specially devoted to study the previous problems when $g = 5$ and

$$S \subset \mathbf{P}^5$$

is the complete intersection of three quadrics. In this case $Q_S = Q^6_S = \mathbf{P}^2$ and $\tilde{Q}^6_S$ is a double covering of $\mathbf{P}^2$ branched on a sextic curve. Let us put, for simplicity of notations,

$$M =: \tilde{Q}^6_S.$$

We will say that M is *the K3–double plane associated to* S. It is well known that:
– *M is a K3 surface,*
– *M is the moduli space $M_S(2; c_1, c_2)$ with $c_1 = H$ and c_2 of degree 4,*
– *S and M are isogenous.*
It is also true that S and M have Néron-Severi lattices of the same rank i.e. they have the same Picard number. This follows immediately from the definition of isogenous K3 surfaces. If S is general the Picard number is one and moreover we have

$$NS(S) = \mathbf{Z}[H] \text{ and } NS(M) = \mathbf{Z}[H']$$

where $H' = \pi^* \mathcal{O}_{\mathbf{P}^2}(1)$ and $\pi : M \to \mathbf{P}^2$ is the above mentioned double covering. In such a case S and M cannot be isomorphic because $c_1(H)^2 = 8$ and $c_1(H')^2 = 2$. On the other hand Mukai has shown the following theorem:

– *Two isogenous K3 surfaces having Picard number $\rho \geq 12$ are isomorphic.*

The inequality $\rho \geq 12$ is sharp. In the second part of this thesis we analyze some examples near the boundary i.e. some K3 surfaces S with Picard number $\rho(S) = 10, 11$. We have considered surfaces S as above which are double coverings of an Enriques surface Y. We give another proof of the following known facts, which is also the subject of [6]:

THEOREM.

1. *If Y is general then $\rho(S) = 10$ and S and M are not isomorphic.*
2. *If Y is a nodal Enriques surface then $\rho(S) = 11$ and S is isomorphic to M.*

We use the lattice–theoretic techniques specially developed by Nikulin, and applied by Morrison and Miranda for studying K3 surfaces with large Picard number. The main difference between the above two cases is due to the intrinsic nature of the corresponding Néron–Severi lattices. More in general we observe that S and M are isomorphic as soon as $NS(S)$ contains a lattice of Todorov type.
Actually, we derive the above results also as a special case of our main result for the case of Picard number (bigger then or) equal to 2, which we are going to describe in the next section. Due to the previous remarks the interesting range for studying the isomorphism problem between and S and M is when

$$2 \ \leq \rho(S) \leq \ 11.$$

3 The main result

Our main new result is the solution of the isomorphism problem when $\rho(S) = 2$, which is one of the result obtained in [7]:

THEOREM. *Let S be a smooth K3 surface which is the base locus of a net of quadrics of $\mathbf{P}^5$ and let M be its associated K3-double plane. Assume $\rho(S) = 2$, then S and M are isomorphic if and only if the following conditions are satisfied:*

1. *$d \equiv 1 \mod 8$, where $-d$ is the determinant of the lattice $NS(S)$,*
2. *one of the two equations $a^2 - db^2 = 8$ or $a^2 + b^2 d = -8$ has integral solutions.*

At first we describe the method and the principal steps of the proof. Then we will make some comments on the geometry and on the numerology behind the statement.
The fundamental tool we have used is the full cohomology lattice

$$H^*(S, \mathbf{Z}) = H^0(S, \mathbf{Z}) \oplus H^2(S, \mathbf{Z}) \oplus H^4(S, \mathbf{Z})$$

endowed with the Mukai product

$$(u, v) =: -(u_0 \cdot v_2 + u_2 \cdot v_0) + u_1 \cdot v_1.$$

Here $u = (u_0, u_1, u_2), v = (v_0, v_1, v_2)$ and $u_i \cdot v_j$ is the usual cup-product in $H^*(S, \mathbf{Z})$. We fix the natural identifications of $H^0(S, \mathbf{Z})$ and $H^4(S, \mathbf{Z})$ with $\mathbf{Z}$, so that

$$(u, v) = -(u_0 v_2 + u_2 v_0) + u_1 \cdot v_1.$$

To any vector $w = (r, H, s)$ one can associate the moduli space $\mathcal{M}_S(w)$ of semistable sheaves $\mathcal{E}$ on S having rank r, determinant H and $\chi(\mathcal{E}) - r = s$. This is the starting point of Mukai's theory, for which we refer to [10]. In particular it is well known that $\dim \mathcal{M}_S(w) = 2 + (w, w)$. We can fix the isotropic vector

$$v = (2, c_1(H), 2)$$

and consider $\mathcal{M}_S(v)$. In this case Mukai remarks that

$$M = \mathcal{M}_S(v)$$

and construct a natural isogeny between S and M which is induced by an algebraic correspondence in $S \times M$. In particular Mukai exhibits a canonical identification

$$H^2(M, \mathbf{Z}) \cong v^\perp / \mathbf{Z}v.$$

This is also an isomorphism between the Hodge structure of $H^2(M, \mathbf{Z})$ and the Hodge structure induced on $v^\perp / \mathbf{Z}v$ by the Hodge structure of $H^*(S, \mathbf{Z})$.
In the following S may have any Picard number. Let L be a lattice and let $x \in L$, by definition

$$\gamma(x) =: \ \min \{ \ xy, \ y \in L, \ xy \geq 0\}$$

Let $H \in NS(S)$ be the class of $\mathcal{O}_S(1)$. Applying some lattice–theoretic results we show that

$$\gamma(H) = 1$$

if S and M are isomorphic. Therefore we will always assume the condition $\gamma(H) = 1$.
Then we observe that, under this condition, S and M are partners i.e. there exists an isometry of transcendental lattices

$$\phi : T(S) \to T(M)$$

which preserves the Hodge structures. In general two K3 surfaces which are partners do not even have isometric Néron–Severi lattices. Moreover, even if the Néron-Severi lattices are isometric, they are not necessarily isomorphic. Some sufficient conditions to be isomorphic, which can be derived by the results in [12], are the following:

(1) the genus of $NS(S)$ is one,
(2) the natural map $O(NS(S)) \to O(A_S)$ is surijective.

Here, $O(NS(S))$ is the orthogonal group of $NS(S)$ and $O(A_S)$ is the orthogonal group of the discriminant of $NS(S)$. We notice that the condition $\gamma(H) = 1$ is always satisfied if $\rho(S) \geq 12$. This essentially follows from the existence of a hyperbolic plane in $NS(S)$. Moreover one can show that the conditions (1) and (2) are also satisfied in this case. Hence it follows $S \cong M$ if $\rho(S) \geq 12$, which is the above mentioned result known to Mukai.

Now we apply the previous general properties and remarks to the case

$$\rho(S) = 2,$$

always assuming $\gamma(H) = 1$. We show that

$$det\ NS(S) \ = \ -d, \quad where\ d \equiv 1 \mod 8\ and\ d > 0.$$

Moreover, under the previous assumptions, $NS(S)$ is uniquely defined by its determinant. More precisely *let*

$$N_d^8$$

be a rank two lattice such that:
(i) N_d^8 *contains a primitive vector* H *with* $H^2 = 8$*,*
(ii) $\det N_d^8 = -d$*, where* $d \equiv 1 \mod 8$ *and* $d > 0$*.*
Then N_d^8 *is unique up to isometries. Moreover all the primitive vectors having self-intersection 8 are in the same orbit of the orthogonal group.*
Very similar properties hold for the Néron-Severi lattice $NS(M)$:

$$det\ NS(M) \ = \ -d, \quad where\ d \equiv 1 \mod 4\ and\ d > 0.$$

Let

$$N_d^2$$

be a rank two lattice such that:
(i) N_d^2 *contains a vector* h *with* $h^2 = 2$*,*
(ii) $\det N_d^8 = -d$*, where* $d \equiv 1 \mod 4$ *and* $d > 0$*.*

Then N_d^2 is unique up to isometries. Moreover all the primitive vectors having self-intersection 2 are in the same orbit of the orthogonal group.

Using these results we deduce that:

S and M have isometric Néron-Severi lattices if and only if the following conditions hold:

(i) $\det NS(S) = \det NS(M) \equiv 1 \mod 8$,

(ii) $NS(S)$ contains a vector of self-intersection 2.

We have already pointed out that two K3 partners may have isomorphic Néron–Severi lattices without being isomorphic. Nevertheless this does not happen in our situation. The proof of this fact is technical and involves more lattice-theoretic constructions. So we omit further details.

We can conclude that S and M are isomorphic iff the previous conditions (i) and (ii) are satisfied. Then we start the description of all cases for which the isomorphism holds.

We assume $d \equiv 1 \mod 8$ and write an appropriate system of generators for $NS(S)$. There is a unique choice of a vector

$$\delta \in NS(S)$$

such that $H\delta = 0$ and $\delta^2 = -8d$. Then one computes that $NS(S)$ is generated by

$$\{H, \quad \delta, \quad \frac{1}{8}(H + \mu\delta)\},$$

where $\mu = 1$ if $d \equiv 1 \mod 16$, $\mu = 3$ if $d \equiv 9 \mod 16$. Equivalently each element z of $NS(S)$ can be written as

$$z = \frac{1}{8}(xH + y\delta),$$

with $x, y \in \mathbf{Z}$ and $\mu x \equiv y \mod 8$. Finally we look to those values of d for which the lattice

$$N_d^8 = NS(S)$$

contains vectors z such that $z^2 = 2$. As above let $z = \frac{1}{8}(xl + y\delta)$, then $z^2 = 2$ if and only if (x, y) are integral odd solutions of the equation

$$x^2 - dy^2 = 16$$

with $x \equiv \pm 4 \mod d$ and $\mu x \equiv y \mod 8$. With some more effort we finally deduce that these conditions are equivalent, for $d \equiv 1 \mod 8$, to the conditions stated in our main theorem. This completes our description of the proof.

We have found two families of values of d:

$$\mathcal{D}_+ = \{\frac{a^2 - 8}{b^2} \in \mathbf{N}, \text{ where } a, b \text{ are odd}\} ,$$

$$\mathcal{D}_- = \{\frac{a^2 + 8}{b^2} \in \mathbf{N}, \text{ where } a, b \text{ are odd}\}.$$

Each of them defines a rank two (hyperbolic) lattice N_d^8 containing vectors z such that $z^2 = 2$. By the surjectivity of the periods map, such a lattice defines a divisor

$$\mathcal{N}_d \subset \mathcal{K}_5$$

parametrizing K3 surfaces S as above which are isomorphic to their associated sextic double plane M. It turns out that $\mathcal{D}_+$ and $\mathcal{D}_-$ are infinite sets. In particular we have found infinitely many divisors $\mathcal{N}_d$ in the moduli space $\mathcal{K}_5$ with the required property.

4 Further comments

We give here some comments which will be subject of further researches.

(1) To add some geometric interpretations to the previous results one can consider the following example, which was the initial motivation to our work. Assume that S contains a line

$$r$$

and that S is sufficently general among the surfaces with this property. Then $NS(S)$ is generated by l and by the class r of r. Moreover

$$NS(S) = N_d^8$$

with $d = 17$ and the equation $x^2 - 17y^2$ admits the integers solutions we want. So $S \cong M$. In this case the isomorphism between S and M can be constructed in a quite simple way.
Let $x \in S \setminus r$ and let P_x be the plane spanned by r and x. Then, it is easy to see that there exists a unique quadric

$$Q \in Q_S$$

which contains P_x. On the other hand the natural double covering

$$\pi : M \to \mathbf{P}^2$$

parametrizes pairs (Q, R), where $Q \in Q_S$ and R is one of the ruling of planes of Q. So we can define a map $\psi : S \to M$ by setting

$$\psi(x) = (Q, R)$$

where R is the ruling of planes containing the element P_x. It turns out that ψ is birational, in particular

$$\psi^* \mathcal{O}_M(1) \cong \mathcal{O}_S(2H - 3r),$$

($H =$ hyperplane section of S). Now recall that $M = \mathcal{M}_S(v)$ is the moduli space of rank two vector bundles $\mathcal{E}$ on S such that $\det \mathcal{E} = \mathcal{O}_S(H)$ and $\deg c_2 = 4$. The point x defines

$$H^1(\mathcal{I}_x(2r - H)) \cong \mathbf{C}.$$

A non zero vector of $H^1(\mathcal{I}_x(2r - H))$ defines an extension

$$0 \to \mathcal{O}_S(r) \to \mathcal{E} \to \mathcal{I}_x(H - r) \to 0.$$

In particular $\mathcal{E}$ is uniquely defined by x and it is semistable. $\mathcal{E}$ defines a point $[\mathcal{E}] \in M$. The map ψ, as one can show, is exactly the map $x \to [\mathcal{E}]$.
Using our knowledge of the lattices N_d^8 we have been able to construct appropriate extensions, and analogous isomorphisms

$$\psi_d : S \to M,$$

for every value of d. A posteriori, this explains in a more geometric way why S and M are isomorphic. In a further work we hope to extend this method to any moduli space

$$M =: M_S(2; c_1, c_2)$$

such that $c_2 = \frac{1}{2}c_1^2$, ($c_1^2 = 2g - 2$, $g \geq 5$). The program would be the following:
– to find lattice theoretic conditions as above so that $S[g - 4]$ and M have isometric Néron-Severi lattices,
– to construct an isomorphism between $S[g - 4]$ and M using, as above, a suitable extension.

In general on a $2n$–dimensional ireeducible compact symplectic Kähler manifold X the self–intersection form on $H^2(X, \mathbf{Z})$ is not quadratic if $n \geq 2$, but it is still possible to define a natural inner product $< \cdot >$ in such a way there exists a rational number q such that

$$\alpha^{2n} = q < \alpha \cdot \alpha >$$

for every $\alpha \in H^2(X, \mathbf{Z})$. The subgroup $NS(S)$ of $H^2(X, \mathbf{Z})$, consisting of the integral cohomology classes perpendicular to the symplectic structure $\omega \in H^0(X, \Omega^2)$ with respect to the inner product $< \cdot >$, endowed with the restriction of $< \cdot >$, is a lattice, the "Néron–Severi lattice" of X. An important result obtained by Mukai in [11] is the following:

– *if two irreducible symplectic Kähler manifolds are birationally equivalent, then their Néron–Severi lattices are isomorphic to each other.*

Note that comparing the above case with the 2–dimensional one, at a certain step of the proof of our result, we need to use Torelli's theorem for K3 surfaces. Such a theorem is not available for $S[g - 4]$ and $M_S(2; c_1, c_2)$, even in the case their Hodge structures are isomorphic.

(2) The result obtained for the Mukai vector $v = (2, H, 2)$, with $H^2 = 8$, in a very similar way, can be derived also for other isotropic Mukai vectors. Only to quote an example, one could consider the Mukai vector

$$v = (2, H, 3)$$

where H is a polarization of degree $H^2 = 12$ on a K3 S, and consider the moduli space

$$M =: \mathcal{M}_S(2, H, 3)$$

which is again a K3 surface. This K3, as showed by Mukai, is endowed by a natural polarization of degree 12, and in general $S \ncong M$, as polarized K3s.

(3) Finally, in [7] it is remarked that the necessary and sufficient conditions found to solve the problem for the Picard number equal to 2, are sufficient when $\rho(S) \geq 3$, and it is not clear if they are still necessary. Up to now it seems that all known examples of higher codimensional locuses where $S \cong M$ lie in the codimension one locuses defined as above.

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